

# The Polynomial Part of the Codimension Growth of Affine PI Algebras

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November 25, 2015

## Abstract

Let  $F$  be a field of characteristic zero and  $W$  be an associative affine  $F$ -algebra satisfying a polynomial identity (PI). The codimension sequence associated to  $W$ ,  $c_n(W)$ , is known to be of the form  $\Theta(cn^t d^n)$ , where  $d$  is the well known (PI) exponent of  $W$ . In this paper we establish an algebraic interpretation of the polynomial part (the constant  $t$ ) by means of Kemer's theory. In particular, we show that in case  $W$  is a *basic* algebra, then  $t = \frac{q-d}{2} + s$ , where  $q$  is the number of simple component in  $W/J(W)$  and  $s + 1$  is the nilpotency degree of  $J(W)$ . Thus proving a conjecture of Giambruno.

## 1 Introduction

Let  $W$  be an affine (i.e. finitely generated) PI  $F$ -algebra over a field  $F$  of characteristic 0. The codimension growth of  $W$  was studied heavily in the last 40 years or so (see for instance the recent books [8, 3]). It provides an important tool for measuring the "size" of the  $T$ -ideal of identities of  $W$  in asymptotic terms. In particular, the exponential part of the asymptotics is key in the classification of varieties of PI algebras. Let us recall briefly some definitions.

A nonzero polynomial  $f(x_1, \dots, x_n) \in F\langle X \rangle$ , in some noncommutative indeterminates  $x_1, \dots, x_n \in X$ , is called a polynomial identity of  $W$  if  $f(w_1, \dots, w_n) = 0$  for all  $w_i \in W$ . If such a polynomial exists then  $W$  is said to be a PI-algebra. The ideal of polynomial identities of  $W$ , denoted  $\text{Id}(W)$ , is closed under endomorphisms  $\phi \in \text{End}_F(F\langle X \rangle)$ . Such ideals are called  $T$ -ideals and are generated, as  $T$ -ideal, by multilinear elements (i.e.  $\cup_{n \in \mathbb{N}} P_n(F)$  with  $P_n(F) = \text{span}_F \{x_{\sigma(1)} \cdots x_{\sigma(n)} \mid \sigma \in S_n\}$ ). Denote by  $c_n(W)$  the dimension  $\dim_F P_n(F)/P_n(F) \cap \text{Id}(W)$ , called the  $n^{\text{th}}$  codimension of  $W$ .

It turns out that the sequence  $c_n(W)$  is exponentially bounded ([11]). In [6] it was proved by Giambruno and Zaicev, confirming a conjecture of Amitsur, that  $\exp(W) = \lim_{n \rightarrow \infty} \sqrt[n]{c_n(W)}$  exists and is an integer called the PI-exponent of  $W$ . Moreover they proved that  $\exp(W)$  is tightly connected with the algebraic structure of  $W$  as long

as  $W \cong A$  is finite dimensional over  $F$  and  $F$  is algebraically closed. Let us state precisely their result. Recall that by Wedderburn-Malcev's theorem any finite dimensional algebra  $A$  over  $F$  can be decomposed into  $A = A_{ss} \oplus J(A)$ , the sum of a maximal semisimple subalgebra which is unique up to isomorphism and its Jacobson radical. Moreover, in case  $F$  is algebraically closed, by Wedderburn's theorem  $A_{ss} \cong A_1 \times \cdots \times A_q$ , where the product is direct and  $A_i \cong M_{d_i}$  is a matrix algebra over  $F$ . Then  $\exp(A) = \max\{\dim_F(A_{i_1} \times \cdots \times A_{i_r}) : \text{where } i_j \neq i_k \text{ for } j \neq k \text{ and } A_{i_1}JA_2 \cdots JA_{i_r} \neq 0\}$ . In order to get Giambruno and Zaicev's result for arbitrary affine algebras (namely  $\exp(W)$  is a nonnegative integer) one needs to invoke the fundamental representability theorem of Kemer.

**Theorem 1.1** (Kemer). *For any affine PI algebra  $W$  there exists a finite dimensional algebra  $A$  such that  $\text{Id}(W) = \text{Id}(A)$ .*

Regev conjectured that the asymptotic behavior of the sequence  $c_n(W)$  should be  $c_n(W) \simeq cn^t d^n$  for some constants  $c \in \mathbb{R}, t \in \frac{1}{2}\mathbb{Z}$  and  $d \in \mathbb{Z}$  (where  $f \simeq g$  means  $\lim \frac{f}{g} = 1$ ). A weakened version of Regev's conjecture was confirmed by Berele and Regev in [5], namely that  $c_n(W)$  is bounded (asymptotically) between the functions

$$c_1 n^t (\exp(W))^n \lesssim c_n(W) \lesssim c_2 n^t (\exp(W))^n \quad (1.1)$$

for some constants  $c_1, c_2 \in \mathbb{R}$  and  $t \in \frac{1}{2}\mathbb{Z}$ . The constant  $t$  is uniquely defined by  $W$ , denoted  $t(W)$  and called the polynomial part of  $W$ . We emphasize that the proof of Berele and Regev, although it proves the existence of the parameter  $t(W)$ , it does not give a formula for its calculation. In this article we present an interpretation (à la Giambruno and Zaicev) of the polynomial part of the codimension growth in case  $W$  is finite dimensional over  $F$ .

In a first step, by work of Kemer on the representability theorem [10], one can reduce the problem to a more specific class of algebras. Namely the so called basic (also called fundamental) algebras. We let  $\text{par}(A) = (\dim_F A_{ss}, \text{nildeg}(J(A)) - 1)$  be the *parameter* of the finite dimensional algebra  $A$ . Here,  $\text{nildeg}(J(A))$  denotes the nilpotency degree of the Jacobson radical  $J(A)$ . Such an algebra  $A$  is said to be basic (or fundamental) if it is not PI equivalent to  $B_1 \times \cdots \times B_l$  where  $\text{par}(B_i) < \text{par}(A)$  for any  $1 \leq i \leq l$ .

The following result was proved by Kemer.

**Theorem 1.2** (Kemer). *Let  $A$  be a finite dimensional algebra over  $F$ . Then there exist basic algebras  $B_1, \dots, B_l$  such that  $A$  is PI-equivalent with  $B_1 \times \cdots \times B_l$ .*

Kemer's result can be used to reduce our problem, namely the interpretation of the polynomial part for finite dimensional algebras, to basic algebras.

**Corollary 1.3.** *Notation as above.  $\exp(A) = \max_{1 \leq i \leq l} \exp(B_i)$  and  $t(A) = \max_j \{t(B_j)\}$  where  $j$  runs over all indices  $j$  for which  $\exp(B_j) = \exp(A)$ .*

*Remark 1.4.* The decomposition into basic algebras (Th. 1.2 above) is a key component of the proof of the representability theorem for affine PI algebras (Th. 1.1). Unlike the

proof of 1.1, the proof of 1.2 is explicit and so Th. 1.2 indeed reduces the calculation of the polynomial part of the codimension asymptotics (of finite dimensional algebras) to their calculation for basic algebras.

For basic algebras Giambruno made the following conjecture.

**Conjecture 1.5** (Giambruno). Let  $A$  be a basic algebra with Wedderburn-Malcev decomposition  $A \cong M_{d_1}(F) \times \cdots \times M_{d_q}(F) \oplus J(A)$ . Then

$$t(A) = \frac{q-d}{2} + s$$

where  $s+1$  equals the nilpotency degree of  $J(A)$ .

In case of matrix algebras  $M_d(F)$  and upper-block triangular matrices  $UT(d_1, \dots, d_q)$  the conjecture is known [12, 7]. Our main result is Theorem 3.29 where we prove the conjecture for an arbitrary basic algebra.

The paper is organized as follows. In section 2 we recall definitions and results on asymptotic PI theory and study basic algebras. The main characterizations of these algebras are also reviewed. We also give some important examples, e.g. we will associate to any finite dimensional algebra some basic algebra  $\mathcal{A}$  which is better understood and yet  $Par(\mathcal{A}) = Par(A)$ . This algebra will be the main tool for the upper bound in section 3.1. Finally in section 3.2 we handle the lower bound and achieve the main result.

*Notation* . Throughout  $F$  will be a field of characteristic 0,  $W$  an affine algebra over  $F$ ,  $A$  a finite dimensional algebra over  $F$ ,  $A_{ss}$  a maximal semisimple subalgebra and  $J(A)$  its radical. By  $\text{nildeg}(J(A))$  we denote the nilpotency degree of  $J(A)$ , i.e. the number such that  $J(A)^l = 0$  but  $J(A)^{l-1} \neq 0$ . Also  $\mathbb{N} = \{n \in \mathbb{Z} \mid n \geq 0\}$ . Finally all polynomials are elements of  $F\langle X \rangle$ , the polynomial ring in countably many, noncommutative variables of  $X$  and  $S_n$  will denote the symmetric group on  $n$  letters.

## 2 Basic algebras

### 2.1 Preliminaries

We start by recalling the general setting. A polynomial  $f(x_1, \dots, x_n) \in F\langle X \rangle$  is called a *polynomial identity* if  $f$  is identically zero on  $W$ , i.e.  $f(w_1, \dots, w_n) = 0$  for all  $(w_1, \dots, w_n) \in W^n$ . The set of all polynomial identities of  $W$  is denoted by  $\text{Id}(W)$ . The latter is not only an ideal of  $F\langle X \rangle$  but is also closed under all endomorphisms  $\phi \in \text{End}_F(F\langle X \rangle)$ . Such ideals are called *T-ideals*.

Clearly  $W$  has the same *T-ideal* of polynomial identities as the algebra  $F\langle X \rangle / \text{Id}(W)$ . This algebra is free in the category of all algebras whose *T-ideal* of identities contains  $\text{Id}(W)$  and is called the *relatively free algebra* of  $W$ . Note that since  $\text{Char}(F) = 0$ , by a

multilinearization argument, all  $f \in \text{Id}(W)$  are consequences of multilinear polynomial identities. Recall,

$$P_n = \text{span}_F \{x_{\sigma(1)} \cdots x_{\sigma(n)} \mid \sigma \in S_n\}$$

are the multilinear polynomial of degree  $n$ . Thus instead of  $F\langle X \rangle / \text{Id}(W)$  one studies  $\bigoplus_n P_n / P_n \cap \text{Id}(W)$  and in particular for all  $n$ , the dimensions

$$c_n(W) = \dim_F P_n(W) = \dim_F P_n / P_n \cap \text{Id}(W)$$

which are called *the  $n^{\text{th}}$ -codimensions of  $W$* .

A first useful feature of codimensions is that they do not vary upon extension of scalars, i.e.  $c_n(W) = c_n(W \otimes_F R)$  where  $R$  a commutative domain. Therefore we may, and will, assume that  $F$  is an algebraically closed field.

It is well known that  $c_n(W)$  is exponentially bounded and Amitsur conjectured that its exponential growth  $\exp(A) = \lim_{n \rightarrow \infty} c_n(W)^{1/n}$  is an integer. This fact was proved by Giambruno and Zaicev in 1998 [6] and for finite dimensional algebras  $A$  they proved an explicit formula connecting it with the algebraic structure of  $A$ . More precisely,

$$\exp(A) = \max\{\dim_F(A_{l_1} \times \cdots \times A_{l_r}) \mid A_{l_1} J(A) A_{l_2} \cdots J(A) A_{l_r} \neq 0, 1 \leq l_1, \dots, l_r \leq q\}$$

where  $A_{l_i} \neq A_{l_j}$  for  $i \neq j$ . Here  $A = A_1 \times \cdots \times A_q \oplus J(A)$  is the *Wedderburn-Malcev decomposition* of  $A$ , where  $J(A)$  is the Jacobson radical of  $A$  and  $A_{ss} = A_1 \times \cdots \times A_q$  is the decomposition into simple factors of a (maximal) semisimple subalgebra of  $A$  which supplements  $J(A)$ .

In 2008 Berele and Regev proved that if one assumes that the codimension sequence  $c_n(W)$  is eventually nondecreasing (i.e. monotonic nondecreasing for large enough  $n$ ) then

$$c_1 n^t d^n \lesssim c_n(W) \lesssim c_2 n^t d^n$$

for constants  $c_1, c_2 \in \mathbb{R}, t \in \frac{1}{2}\mathbb{Z}$  and  $d \in \mathbb{Z}$ . Moreover if  $W$  is unital (where clearly, the sequence is nondecreasing)  $c_1 = c_2$  and thus confirming in this case a conjecture of Regev. Recently, in [9], Giambruno and Zaicev proved that the sequence of codimensions is indeed eventually nondecreasing and hence establishing the above asymptotic inequality. The numbers  $d$  and  $t$  are called the *exponent* and *polynomial part* of  $W$  and are denoted  $\exp(A)$  and  $t(W)$  respectively.

As mentioned in the introduction at PI-theoretical level there is no difference between affine PI algebras and finite dimensional algebras [10]. A detailed account of Kemer's representability theorem for affine PI algebras can be found in [1].

Recall now the definition of a basic (or fundamental) algebra.

**Definition 2.1.** The *parameter* of  $A$  is the pair

$$\text{par}(A) = (d = \dim_F A_{ss}, \text{nildeg}(J(A)) - 1).$$

We say that  $A$  is *basic* if it is not possible to find an algebra  $B$ , PI equivalent to  $A$ , with  $A \cong B_1 \times \cdots \times B_l$  and such that  $\text{par}(B_i) < \text{par}(A)$  for any  $1 \leq i \leq l$ .

By Giambruno and Zaicev's theory on codimensions we have that if  $A$  is basic with Kemer index  $(d, s)$  then  $\exp(A) = d = \dim(A_{ss})$ . Note that if  $q$  denotes the number of simple components of  $A_{ss}$ , then  $q \leq s = \text{nildeg}(J(A)) - 1$ .

As mentioned in the introduction, the polynomial part of the codimension of a finite dimensional algebra (and by Kemer's theory of an affine PI algebra) is determined by its value on basic algebras (see [4, Lemma 3.2]). For basic algebras Giambruno conjectured the following.

**Conjecture 2.2** (Giambruno). Let  $A$  be a basic algebra and  $A \cong A_1 \times \cdots \times A_q \oplus J(A)$  with  $A_i = M_{d_i}(F)$ . Then

$$t(A) = t(A_1) + \cdots + t(A_q) + (\text{nildeg}(J(A)) - 1).$$

Note that by Regev's result [12],  $t(M_d(F)) = \frac{1-d^2}{2}$  and hence Giambruno's conjecture takes the form

$$t(A) = (q - d)/2 + (\text{nildeg}(J(A)) - 1)$$

**Example 2.3.** Let  $A = UT(d_1, \dots, d_q)$  be the block upper triangular matrices. This is a basic algebra. It was proved by Giambruno and Zaicev [7], applying Lewin's theorem, that  $\text{Id}(A) = \text{Id}(M_{d_1}(F)) \cdots \text{Id}(M_{d_q}(F))$ . Now by a result of Berele and Regev [4] the exact asymptotics of  $c_n(A)$  can be expressed as function of  $d_i$  for  $1 \leq i \leq q$ . Precise formulas for the exponential and polynomial parts are given by  $\exp(A) = d = d_1^2 + \cdots + d_q^2$  and  $t(A) = -\frac{1}{2}(\sum_{i=1}^q d_i^2 - 3q + 2) = \frac{q-d}{2} + (q-1)$ . Clearly  $\text{nildeg}(J(A)) = q$ , thus  $UT(d_1, \dots, d_q)$  indeed satisfies the conjecture.

In the next section we present equivalent conditions for an algebra to be basic. In particular, we will explain that basic algebras are minimal models for a given Kemer index.

## 2.2 On structure of Basic algebras

As always in this article  $A \cong A_1 \times \cdots \times A_q \oplus J(A)$  with  $A_i \cong M_{d_i}(F)$ . Thus  $d_i^2 = \dim A_i$ .

**Definition 2.4** (Kemer index). Define for any  $\nu \in \mathbb{N}$  the set

$$\Delta_\nu = \{r \in \mathbb{N} \cup \{0\} \mid \exists p(X) \notin \text{Id}(A) \text{ alternating in } \nu \text{ disjoint sets of size } r\}.$$

Clearly if  $\nu \leq \gamma$ , then  $\max \Delta_\nu \geq \max \Delta_\gamma$ . Let  $d = \lim_\nu \max(\Delta_\nu)$ . Further define the set

$$S_\nu^d = \{j \in \mathbb{N} \cup \{0\} \mid \exists p(X) \notin \text{Id}(A) \text{ alternating in } \nu \text{ disjoint sets of size } d \text{ and alternating in } j \text{ sets of size } d+1\}$$

and the number

$$s = \lim_\nu \max S_\nu^d$$

where  $\max S_\gamma^d \leq \max S_\nu^d$  if  $\gamma \geq \nu$ .

The tuple  $\kappa_A = (d, s)$  is called the *Kemer index* of  $A$ .

The alternating sets of size  $d$  are called *small sets* and the sets of size  $d + 1$  *big sets*. In other words, there exist nonidentities in an arbitrary number of alternating small sets, but only a finite number of big sets (actually  $s$  big sets). Note that  $d$  and  $s$  are finite. Indeed,  $d$  is finite because  $\max \Delta_\nu \leq \dim A$  and  $s$  is finite due to the definition of  $d$ .

For a general finite dimensional algebra one has  $\kappa_A = (d, s) \leq \text{par}(A)$  in the lexicographic order [1]. Equality characterizes basic algebras.

**Theorem 2.5.** *A finite dimensional algebra  $A$  is basic if and only if  $\kappa_A = \text{par}(A)$ .*

An important consequence of this theorem is the existence of Kemer polynomials. These are “extremal” nonidentities which vanish either if not all simple components  $A_i$  are represented among the substitutions or if *less* than  $J(A) - 1$  variables are substituted by radical elements.

**Definition 2.6.** Let  $\nu_0$  be a number where  $\max \Delta_{\nu_0} = d$  and  $\max S_{\nu_0}^d = s$ . Then a multilinear polynomial  $f$  is called a *Kemer polynomial* of  $A$  if  $f \notin \text{Id}(A)$  has at least  $\nu_0$  small sets (cardinality  $d$ ) and exactly  $s$  big sets (cardinality  $d + 1$ ).

## 2.3 Examples of Basic algebras

### 2.3.1 Well known examples

The matrix algebra  $M_d(F)$  has Kemer index  $(d^2, 0)$  and thus is basic. To see this, note that any multilinear polynomial alternating on  $d^2 + 1$  variables vanishes on  $M_d(F)$ . Now recall the  $n$ th Capelli polynomial

$$\text{cap}_n(X; Y) = \sum_{\sigma \in S_n} \text{sign}(\sigma) y_1 x_{\sigma(1)} \cdots y_n x_{\sigma(n)} y_{n+1}.$$

It is well known, [8, Prop. 1.7.1], that  $\text{cap}_{d^2+1}(X; Y) \in \text{Id}(M_d(F))$  but  $\text{cap}_{d^2}(X; Y) \notin \text{Id}(M_d(F))$  and moreover all diagonal elementary matrices can be realized as a nonzero evaluation. Therefore for any  $\mu \in \mathbb{N}$  the polynomial

$$\text{Cap}_{d^2}(X_1, \dots, X_\mu; Y_1, \dots, Y_\mu) = \prod_{i=1}^{\mu} \text{cap}_{d^2}(X_i, Y_i)$$

where  $X_i = \{x_{i,1}, \dots, x_{i,d^2}\}$  is a Kemer polynomial of  $M_d(F)$ , proving that  $\kappa_{M_d(F)} = (d^2, 0)$ .

The next natural and important example is the algebra of upper block triangular matrices  $UT(d_1, \dots, d_q)$  for positive integers  $d_1, \dots, d_q$ . This is the subalgebra of  $M_{d_1+\dots+d_q}(F)$  consisting of the matrices

$$\begin{pmatrix} M_{d_1}(F) & & * \\ 0 & \ddots & \\ \vdots & & \\ 0 & \cdots & 0 & M_{d_q}(F) \end{pmatrix}.$$

This is a basic algebra with Kemer index  $(d, q - 1)$ , where  $d = d_1^2 + \dots + d_q^2$ . Indeed, it is well known that  $UT(d_1, \dots, d_q)$  has exponent  $d$  and hence its Kemer index has the form  $\kappa = (d, s)$ . Moreover, since the nilpotency index of  $UT(d_1, \dots, d_q)$  is  $q - 1$  we have  $s \leq q - 1$ . In order to complete the proof we'll construct Kemer polynomials with arbitrary many small sets of cardinality  $d$  and precisely  $q - 1$  sets of cardinality  $d + 1$ . We start with the construction of polynomials with arbitrary many small sets of cardinality  $d$ . For the simple component  $M_{d_i}(F)$ ,  $i = 1, \dots, q$ , we consider the polynomial  $\text{Cap}_{d_i^2}(X_{i,1}, \dots, X_{i,\mu}; Y_{i,1}, \dots, Y_{i,\mu})$  and their product bridged by  $w_1, \dots, w_{q-1}$

$$\text{Cap}_{d_1^2}(X_{1,1}, \dots, X_{1,\mu}; Y_{1,1}, \dots, Y_{1,\mu}) \times w_1 \dots \times w_{q-1} \times \text{Cap}_{d_q^2}(X_{q,1}, \dots, X_{q,\mu}; Y_{q,1}, \dots, Y_{q,\mu}).$$

which we denote by  $\text{Cap}_{d_1^2, \dots, d_q^2}(X_{i,j}; Y_{i,j}, i = 1, \dots, q, j = 1, \dots, \mu, W)$  or

for short  $\text{Cap}_{d_1^2, \dots, d_q^2}(X_{i,j}; Y_{i,j}, W)$ .

Now for  $j = 1, \dots, \mu$  we alternate in the polynomial above the sets  $X_{1,j}, \dots, X_{q,j}$  and obtain a polynomial which we denote by  $f_{1,\mu}(X_{i,j}; Y_{i,j}, W)$ . Finally we construct  $q - 1$  big sets by alternating any  $w_j$  with the set  $X_j = X_{1,j} \cup \dots \cup X_{q,j}$ . The result is a polynomial  $f_{2,\mu}$  which alternates on  $\mu - (q - 1)$  small sets of cardinality  $d$  and precisely  $q - 1$  big sets of cardinality  $d + 1$ . We leave the reader the task to show that  $f_{1,\mu}$  and  $f_{2,\mu}$  are nonidentities of  $UT(d_1, \dots, d_q)$  by presenting nonzero evaluations (variables of  $W$  are evaluated on radical elements). Our construction of  $f_{2,\mu}$  shows that  $\kappa$ , the Kemer index of  $UT(d_1, \dots, d_q)$ , satisfies  $\kappa_{UT(d_1, \dots, d_q)} \geq (d, q - 1)$ . On the other hand  $\kappa_{UT(d_1, \dots, d_q)} \leq \text{Par}_{UT(d_1, \dots, d_q)} = (d, q - 1)$  and hence  $\kappa = \text{Par}$ . This shows  $UT(d_1, \dots, d_q)$  is basic and  $f_2$  is Kemer.

### 2.3.2 The associated basic algebra

To any algebra  $A$  one can associate for any  $u \in \mathbb{N}$  the algebra

$$\mathcal{A}_u := \frac{A_{ss} * F\{x_1, \dots, x_r\}}{\langle x_1, \dots, x_r \rangle_{A_{ss} * F\{x_1, \dots, x_r\}}^u}.$$

We are interested in the case where  $u = \text{nildeg}(J(A))$ . For convenience we denote  $\mathcal{A} = \mathcal{A}_{\text{nildeg}(J(A))}$ .

**Lemma 2.7.** *The algebra  $\mathcal{A}$  is finite dimensional. Moreover, if the algebra  $A$  is basic then  $\mathcal{A}$  is also basic.*

*Proof.* By definition of  $\mathcal{A}$  we see that multilinear monomials are zero if the number of variables exceeds  $u - 1$  and hence the number of configurations of the  $x_i$ 's is finite. Next, the variables may be bridged by semisimple elements which belong to a finite dimensional algebra proving the first part of the lemma. For the second part, note that  $A_{ss}$  is a maximal semisimple subalgebra which supplements the radical  $J(\mathcal{A})$ . Moreover, the radical is generated by the variables  $x_i$  and its nilpotency degree equals  $\text{nildeg}(J(A))$ . It follows that the  $\text{par}((A)) = (d, u - 1)$ . But the algebra  $A$  is a quotient

of  $\mathcal{A}$  and hence its Kemer index exceeds the Kemer index of  $A$ . This implies the Kemer index of  $\mathcal{A}$  equals  $\text{par}((A)) = (d, u - 1)$  and the result follows.  $\square$

### 3 Giambruno Conjecture

During the whole section we consider a basic  $F$ -algebra  $A$  whose Wedderburn-Malcev decomposition is given by

$$A = A_{ss} \oplus J(A),$$

where  $A_{ss} = A_1 \times \cdots \times A_q$  is a product of matrix algebras, say  $A_i \cong M_{d_i}(F)$ ,  $i = 1, \dots, q$ . Further denote  $\kappa_A = (d, s)$  the Kemer index of  $A$ . In particular by Theorem 2.5,  $d = d_1^2 + \cdots + d_q^2$  and  $s = \text{nildeg}(J(A)) - 1$  the nilpotency index of  $J(A)$  minus 1.

The proof consists of two parts, namely we show  $(q - d)/2 + s - 1$  bounds from above and from below  $t(A)$ . For the upper bound observe that  $\text{id}(\mathcal{A}) \subseteq \text{id}(A)$ . Also the algebras  $A$  and  $\mathcal{A}$  have the same Kemer index and moreover have isomorphic semisimple subalgebras supplementing the corresponding radicals. In particular they have the same exponent. It follows that  $t(A) \leq t(\mathcal{A})$  and hence it is sufficient to show  $t(\mathcal{A}) \leq (q - d)/2 + s - 1$ .

#### 3.1 Upper bound

As remarked above we need to show  $t(\mathcal{A}) \leq (q - d)/2 + s - 1$  where  $\mathcal{A}$  is the basic algebra

$$\mathcal{A} = \frac{A_{ss} * F\{b_1, \dots, b_{\dim_F J(A)}\}}{\langle b_1, \dots, b_{\dim_F J(A)} \rangle^{s+1}}.$$

##### 3.1.1 The reductions

First, let us describe a preferable basis for  $\mathcal{A}$ : For  $1 \leq l \leq q$  we denote the matrix units of  $A_l$  by  $e_{j_1, j_2}(A_l)$  and  $e_{j_1}(A_l) = e_{j_1, j_1}(A_l)$ . Next, define

$$W_{i,j}(A_k, A_{k'}) = \{e_{i, j_0}(A_k)b_{l_0}e_{i_1, j_1}(A_{k_1})b_{l_1} \cdots e_{i_{s'}, j_{s'}}(A_{k_{s'}})b_{l_{s'}}e_{i_{s'+1}, j}(A_{k'}) | 0 \leq s' \leq s\}.$$

The union of all  $W_{i,j}(A_{k_1}, A_{k_2})$  is denoted by  $W$ . Thus, a basis for  $\mathcal{A}$  is the set

$$\{e_{j_1, j_2}(A_l) | 1 \leq l \leq q, 1 \leq j_1, j_2 \leq d_l\} \cup W.$$

Since  $\mathcal{A}$  is a finite dimensional algebra by proposition 2.7, we can identify its relatively free algebra  $F\langle X_i | i \in \mathbb{N} \rangle / \text{Id}(\mathcal{A})$  with a subalgebra of

$$\mathcal{A}_K = \mathcal{A} \otimes_F K,$$



where  $K$  is the (commutative) polynomial ring

$$K = F \left[ \theta_{j_1, j_2}^{(i)}(A_l), \theta^{(i)}(w) \mid i \in \mathbb{N}, 1 \leq l \leq q, 1 \leq j_1, j_2 \leq d_l, w \in W \right].$$

Let us explicitly make the identification. Write,

$$X_i(A_k) = \sum_{a_1, a_2} \theta_{a_1, a_2}^{(i)}(A_k) e_{a_1, a_2}(A_k).$$

Then the variable  $X_i$  of the relatively free algebra of  $\mathcal{A}$  is identified with

$$\sum_{k=1}^q X_i(A_k) + \sum_{w \in W} \underbrace{\theta^{(i)}(w)}_{X_i(w)} w = \sum_{\Sigma} X_i(\Sigma) \in \mathcal{A}_K,$$

where  $\Sigma$  is a symbol which runs over the set  $\mathbf{Symb} = W \cup (\mathbf{SimComp} := \{A_1, \dots, A_q\})$ . As a result of this identification, the spaces  $P_n(\mathcal{A})$  are considered as subspaces of  $\mathcal{A}_K$ .

We decompose  $P_n(\mathcal{A})$  into subspaces as follows. Consider a monomial  $X_{\sigma(1)} \cdots X_{\sigma(n)} \in P_n(\mathcal{A})$ , where  $\sigma \in S_n$ . Clearly the identification of the monomial in  $\mathcal{A}_K$  is equal to the sum

$$\sum_{\Sigma_1, \dots, \Sigma_n \in \mathbf{Symb}} X_{\sigma(1)}(\Sigma_1) \cdots X_{\sigma(n)}(\Sigma_n). \quad (3.1)$$

Note that:

1.  $X_i(A_k)X_j(A_{k'}) = 0$  if  $k \neq k'$ .
2. If more than  $s$  symbols from  $\Sigma_1, \dots, \Sigma_n$  are radical (that is, from  $W$ ), then

$$X_{\sigma(1)}(\Sigma_1) \cdots X_{\sigma(n)}(\Sigma_n) = 0.$$

This leads to the following definition.

**Definition 3.1.** A sequence  $\vec{p} = (p_1, \dots, p_n)$  of symbols in  $\mathbf{Symb}$  is called a *path* (of length  $n$ ) if the following two properties are satisfied:

1. If  $p_i, p_{i+1} \in \mathbf{SimComp}$ , then  $p_i = p_{i+1}$ .
2. No more than  $s$  symbols (from  $p_1, \dots, p_n$ ) are in  $W$ .

Furthermore, suppose  $\vec{p} = (A_{k_1}, \dots, A_{k_1}, w_1, A_{k_2}, \dots, A_{k_2}, w_2, \dots, w_{s'}, A_{k_{s'+1}}, \dots, A_{k_{s'+1}})$ , then the *path structure* of  $\vec{p}$ ,  $\mathbf{struc}(\vec{p})$ , is defined to be the sequence  $(A_{k_1}, w_1, A_{k_2}, w_2, \dots, w_{s'}, A_{k_{s'+1}})$  (i.e. we record the simple components with no repetitions and the radical elements). Two paths  $\vec{p}_1, \vec{p}_2$  (of the same length) are called *equivalent* (denoted by  $\vec{p}_1 \sim \vec{p}_2$ ) if they have the same path structure.

The set of all paths of length  $n$  is denoted by  $\mathbf{Path}_n$  and the set of all equivalent classes of paths of length  $n$  is denoted by  $\mathbf{Path}_n / \sim$ .

**Definition 3.2.** For a given path  $\vec{p}$  we denote the number of appearances of a symbol  $\Sigma$  from **Symb** by  $\vec{p}(\Sigma)$ .

By definition, equation (3.1) can now be rewritten as

$$\sum_{\vec{p}=(p_1,\dots,p_n)\in\mathbf{Path}_n} X_{\sigma(1)}(p_1)\cdots X_{\sigma(n)}(p_n) = \sum_{[\vec{p}]\in\mathbf{Path}_n/\sim} \left( \sum_{\vec{p}=(p_1,\dots,p_n)\in[\vec{p}] } X_{\sigma(1)}(p_1)\cdots X_{\sigma(n)}(p_n) \right).$$

**Definition 3.3.** For a fixed  $\vec{p}$  denote by  $P_{\vec{p}}(\mathcal{A})$  the  $F$ -linear span of all monomials  $X_{\sigma(1)}(p_1)\cdots X_{\sigma(n)}(p_n)$ , where  $\sigma$  varies over  $S_n$ . Furthermore,  $P_{[\vec{p}]}(\mathcal{A})$  denotes the sum of all  $P_{\vec{p}}(\mathcal{A})$  such that  $\vec{p} \sim \vec{p}_1$ .

We have

**Lemma 3.4.** *The space  $P_n(\mathcal{A})$  is embedded in*

$$\bigoplus_{\vec{p}\in\mathbf{Path}_n} P_{\vec{p}}(\mathcal{A}) = \bigoplus_{[\vec{p}]\in\mathbf{Path}_n/\sim} P_{[\vec{p}]}(\mathcal{A}).$$

As a result,

$$c_n(A) \leq \sum_{[\vec{p}]\in\mathbf{Path}_n/\sim} \dim_F P_{[\vec{p}]}(\mathcal{A}).$$

Moreover, the size of the set  $\mathbf{Path}_n/\sim$  is bounded by a constant independent of  $n$ .

*Proof.* Only the last part requires an explanation. Indeed, the size of  $\mathbf{Path}_n/\sim$  is bounded from above by the number of sequences of length at most  $2s+1$  whose elements are taken from the finite set **Symb**. So the constant can be taken to be

$$\sum_{t=1}^{2s+1} |\mathbf{Symb}|^t.$$

□

*Remark 3.5.* By the previous Lemma, it is sufficient to show that  $\dim_F P_{[\vec{p}]}(\mathcal{A})$  is bounded from above by  $Cn^{\frac{q-d}{2}+s}d^n$ , where  $C$  is some constant.

We intend to decompose each  $P_{[\vec{p}]}(\mathcal{A})$  into a (direct) sum of some special subspaces.

**Definition 3.6.** Let  $\vec{p}$  be a path and  $Z = X_{\sigma(1)}(p_1)\cdots X_{\sigma(n)}(p_n)$  be a monomial in  $P_{\vec{p}}(\mathcal{A})$ . For  $1 \leq l \leq q$  we denote by  $\mathbf{ind}_l(Z)$  the set of all indices  $\sigma(u)$  (here  $1 \leq u \leq n$ ) for which  $p_u = A_l$ .

Furthermore, we denote by  $\mathbf{seq}_{rad}(Z)$  the sequence of indices  $(\sigma(i_1), \dots, \sigma(i_{s'}))$  for which

1.  $p_{i_u} \in W$  for every  $1 \leq u \leq s'$ .

2.  $i_1 < \dots < i_{s'}.$
3.  $\{\sigma(i_1), \dots, \sigma(i_{s'})\} \cup \mathbf{ind}_1(Z) \cup \dots \cup \mathbf{ind}_q(Z) = \{1, \dots, n\}$  (that is  $\mathbf{seq}_{rad}(Z)$  consists of all the indices whose corresponding variables take values in the radical).

Finally, we set  $\overrightarrow{\mathbf{ind}}(Z) = (\mathbf{ind}_1(Z), \dots, \mathbf{ind}_q(Z); \mathbf{seq}_{rad}(Z)).$

**Definition 3.7.** Two monomials  $Z_1$  and  $Z_2$  in  $P_{[\vec{p}]}(\mathcal{A})$  are *equivalent* (or  $Z_1 \sim Z_2$ ) if  $\overrightarrow{\mathbf{ind}}(Z_1) = \overrightarrow{\mathbf{ind}}(Z_2)$ . The set of all equivalence classes corresponding to this relation is denoted by  $\mathbf{Mon}_{[\vec{p}]} / \sim$ , where  $\mathbf{Mon}_{[\vec{p}]}$  is the set of monomials in  $P_{[\vec{p}]}(\mathcal{A})$ . Furthermore,  $P_{[Z]}(\mathcal{A}) (\subseteq P_{[\vec{p}]}(\mathcal{A}))$  denotes the  $F$ -span of all monomials in  $P_{[\vec{p}]}(\mathcal{A})$  which are equivalent to  $Z$ . (Order the definitions by putting **Mon** for the set of monomials and  $P$  for the corresponding  $F$ -span of the elements).

**Lemma 3.8.** *The following hold:*

1.  $P_{[\vec{p}]}(\mathcal{A})$  is equal to

$$\bigoplus_{[Z] \in \mathbf{Mon}_{[\vec{p}]} / \sim} P_{[Z]}(\mathcal{A}).$$

2. Denote by  $\mathbf{Mon}_{[\vec{p}]}(n_1, \dots, n_q) / \sim$  the subset of  $\mathbf{Mon}_{[\vec{p}]} / \sim$  consisting of all  $[Z]$  for which the corresponding path  $\vec{p}$  satisfies  $n_1 = |\mathbf{ind}_1(Z)|, \dots, n_q = |\mathbf{ind}_q(Z)|$ . Then,  $\mathbf{Mon}_{[\vec{p}]} / \sim$  is equal to the (disjoint) union

$$\bigcup_{n_1 + \dots + n_q = n - s'} \mathbf{Mon}_{[\vec{p}]}(n_1, \dots, n_q) / \sim.$$

3. The size of  $\mathbf{Mon}_{[\vec{p}]}(n_1, \dots, n_q) / \sim$  is bounded from above by

$$n^{s'} \cdot \binom{n - s'}{n_1, \dots, n_q},$$

where  $s' = |\mathbf{seq}_{rad}(Z)|$  the number of symbols from  $W$  in  $\vec{p}$ .

*Proof.* Only the third part requires a proof: There are  $s'! \cdot \binom{n}{s'}$  options to choose and order  $s'$  indices from the set  $\{1, \dots, n\}$ , i.e. there are  $s'! \cdot \binom{n}{s'}$  ways to choose  $\mathbf{seq}_{rad}$  for a fixed  $1 \leq s' \leq s$ . From the remaining  $n - s'$  indices there are  $\binom{n - s'}{n_1, \dots, n_q}$  options to choose  $n_1$  which will correspond to  $A_1, \dots, n_q$  which will correspond to  $A_q$ . Finally, it is clear that

$$s'! \cdot \binom{n}{s'} \binom{n - s'}{n_1, \dots, n_q} \leq n^{s'} \binom{n - s'}{n_1, \dots, n_q}.$$

□

**Definition 3.9.** For  $\vec{i} = (i_1, \dots, i_l)$  we denote the product  $X_{i_1}(A_j) \cdots X_{i_l}(A_j)$  by  $\mathbf{X}_{\vec{i}}(A_j)$ .

Consider monomials in  $P_{[Z]}(\mathcal{A}) (\subseteq P_{[\vec{p}]}(\mathcal{A}))$  of the form

$$\mathbf{X}_{\vec{i}_1}^{\vec{p}}(A_{k_1})X_{\nu_1}(w_1)\mathbf{X}_{\vec{i}_2}^{\vec{p}}(A_{k_2})X_{\nu_2}(w_2)\cdots\mathbf{X}_{\vec{i}_{s'+1}}^{\vec{p}}(A_{k_{s'+1}}),$$

namely, monomials with  $\mathbf{struc}(\vec{p}) = (A_{k_1}, w_1, A_{k_2}, w_2, \dots, A_{k_{s'+1}})$ ,  $\bigcup_{\alpha: k_\alpha=l} \text{Set}_{\vec{i}_\alpha} = \mathbf{ind}_l(Z)$ , where  $\text{Set}_x$  consists of all indices appearing in the vector  $x$ , and  $\mathbf{seq}_{rad}(Z) = (\nu_1, \dots, \nu_{s'})$ .

*Remark 3.10.* It is important to stress that there exist other type of monomials, namely monomials where some radical elements are adjacent or monomials which start or end by radical elements. As it will be clear below, these “degenerate” monomials are easier to treat.

Consider the spaces  $P_{[Z]}^{j_0, j_{s'+1}}(\mathcal{A}) := e_{j_0}(A_{k_1})P_{[Z]}(\mathcal{A})e_{j_{s'+1}}(A_{k_{s'+1}}) (\subseteq \mathcal{A}_K)$ , where  $1 \leq j_0 \leq d_{k_1}$  and  $1 \leq j_{s'+1} \leq d_{k_{s'+1}}$  (recall our notation  $e_j(B)$ , the diagonal matrix  $e_{j,j}$  in the matrix algebra  $B$ ). Since every element  $f$  of  $P_{[Z]}(\mathcal{A})$  can be written as the sum

$$f = 1(A_{k_1}) \cdot f \cdot 1(A_{k_{s'+1}}) = \sum_{\substack{1 \leq j_0 \leq d_{k_1} \\ 1 \leq j_{s'+1} \leq d_{k_{s'+1}}}} e_{j_0}(A_{k_1}) \cdot f \cdot e_{j_{s'+1}}(A_{k_{s'+1}}),$$

we obtain an injective map

$$P_{[Z]}(\mathcal{A}) \rightarrow \bigoplus_{\substack{1 \leq j_0 \leq d_{k_1} \\ 1 \leq j_{s'+1} \leq d_{k_{s'+1}}}} P_{[Z]}^{j_0, j_{s'+1}}(\mathcal{A}).$$

So we proved:

**Lemma 3.11.**  $\dim_F P_{[Z]}(\mathcal{A}) \leq \sum_{j_0, j_{s'+1}} \dim_F P_{[Z]}^{j_0, j_{s'+1}}(\mathcal{A}).$

As a result of this observation we will fix also the indices  $j_0, j_{s'+1}$  and work in the space  $P_{[Z]}^{j_0, j_{s'+1}}(\mathcal{A})$ . In Lemma 3.12 we describe the asymptotics of  $\dim_F P_{[Z]}^{j_0, j_{s'+1}}(\mathcal{A})$  and subsequently we will sum up all this kind of spaces by taking precisely into account how many of them we have.

### 3.1.2 The key lemma and upper bound

In order to proceed let us summarize our set up. We fix the path  $\vec{p}$  and numbers  $n_1, \dots, n_q$ . Also  $[Z]$  such that  $|\mathbf{ind}_j(Z)| = n_j$  is fixed.

In order to be able to carry out manipulations in the vector space  $P_{[Z]}^{j_0, j_{s'+1}}(\mathcal{A})$  we introduce the following notations:

$$\theta_{a_1, \dots, a_{l+1}}^{X_1, \dots, X_l}(A_k) := \theta_{a_1, a_2}^{(X_1)}(A_k) \theta_{a_2, a_3}^{(X_2)}(A_k) \cdots \theta_{a_l, a_{l+1}}^{(X_l)}(A_k)$$

and

$$\theta_{i|j}^{X_1, \dots, X_l}(A_k) := \sum_{a_2, \dots, a_l} \theta_{a_1=i, a_2, \dots, a_l, a_{l+1}=j}^{X_1, \dots, X_l}(A_k).$$

Note that we changed slightly the notation we introduced above by replacing  $\theta_{a_k, a_r}^{(l)}$  with  $\theta_{a_k, a_r}^{(X_l)}$ .

Furthermore, if  $\vec{\nu} = (1, \dots, l)$  we simply write

$$\theta_{i|j}^{X_{\vec{\nu}}}(A_k) = \theta_{i|j}^{X_1, \dots, X_l}(A_k).$$

The next Lemma is key.

**Lemma 3.12.** *The following hold.*

1. For  $w_1 \in W_{-,i}(A_-, A_k), w_2 \in W_{j,-}(A_k, A_-)$  we have

$$w_1 \mathbf{X}_{\vec{\nu}}(A_k) w_2 = w_1 \theta_{i|j}^{X_{\vec{\nu}}}(A_k) w_2.$$

2. For  $w_1 \in W_{j_1, \tilde{j}_1}(A_{k_1}, A_{k_2}), \dots, w_{s'} \in W_{j_{s'}, \tilde{j}_{s'}}(A_{k_{s'}}, A_{k_{s'+1}})$  we have

$$\underbrace{e_{\tilde{j}_0=j_0}(A_{k_0})}_{w_0} \left( \mathbf{X}_{\vec{\nu}_1}(A_{k_1}) X_{i_1}(w_1) \mathbf{X}_{\vec{\nu}_2}(A_{k_2}) X_{i_2}(w_2) \cdots \mathbf{X}_{\vec{\nu}_{s'+1}}(A_{k_{s'+1}}) \right) \underbrace{e_{\tilde{j}_{s'+1}=j_{s'+1}}(A_{k_{s'+1}})}_{w_{s'+1}}$$

equals to

$$\theta^{(i_1)}(w_1) \cdots \theta^{(i_{s'})}(w_{s'}) \left( \prod_{l=1}^q \prod_{\alpha: k_\alpha=l} \theta_{\tilde{j}_{\alpha-1}|j_\alpha}^{X_{\vec{\nu}_\alpha}}(A_l) \right) w_0 \cdots w_{s'+1}.$$

3. For every  $l$ , consider the variables  $Y_{1,l}, \dots, Y_{v-1,l}$ , where  $v = v(l)$  is the number of appearances of  $A_l$  in  $\mathbf{struc}([\vec{p}])$ . Then the expression

$$\prod_{\alpha: k_\alpha=l} \theta_{\tilde{j}_{\alpha-1}|j_\alpha}^{X_{\vec{\nu}_\alpha}}(A_l)$$

is equal to

$$\pi_l \left( \theta_{\tilde{r}_l=\tilde{j}_{\alpha(1)-1}|j_{\alpha(v)}=r_l}^{(X_{\vec{\nu}_{\alpha(1)}}^{Y_{1,l}}, Y_{1,l}, X_{\vec{\nu}_{\alpha(2)}}^{Y_{2,l}, \dots, Y_{v-1,l}}, X_{\vec{\nu}_{\alpha(v)}}^{Y_{v-1,l}})=\widehat{X_{A_l}}}(A_l) \right),$$

where  $\{\alpha(1), \dots, \alpha(v)\} = \{\alpha \mid k_\alpha = l\}$ ,  $\alpha(1) < \dots < \alpha(v)$ , and  $\pi_l$  is the evaluation  $Y_{1,l} \rightarrow e_{\tilde{j}_{\alpha(1)-1}, j_{\alpha(2)}}(A_l), \dots, Y_{v-1,l} \rightarrow e_{\tilde{j}_{\alpha(v-1)-1}, j_{\alpha(v)}}(A_l)$ .

4. The map

$$\Psi : P_{\widehat{X_{A_1}}}(A_1) \otimes \cdots \otimes P_{\widehat{X_{A_q}}}(A_q) \rightarrow P_{[Z]}^{j_0, j_{s'+1}}(\mathcal{A})$$

which sends

$$f_1 \otimes \cdots \otimes f_q$$

to

$$\theta^{(i_1)}(w_1) \cdots \theta^{(i_{s'})}(w_{s'}) \left( \prod_{l=1}^q \pi_l \circ \psi_{\tilde{r}_l, r_l}(f_l) \right) \cdot w_0 \cdots w_{s'+1},$$

where  $\psi_{\tilde{r}_l, r_l}(a)$  is the coefficient of  $e_{\tilde{r}_l, r_l}$ , is a well defined surjection.

5.  $\dim_F P_{[Z]}^{j_0, j_{s'}+1}(\mathcal{A}) \leq c_{n_1+s}(A_1) \cdots c_{n_q+s}(A_q) \leq C \cdot c_{n_1}(A_1) \cdots c_{n_q}(A_q)$ , where  $C$  is a constant which is independent of  $n_1, \dots, n_q$ .

*Proof.* The first part follows easily from the definitions. The second part follows from part (1). The third part is trivial. The fourth part follows immediately from the third part.

We are left with part (5): by part (4)

$$\dim_F P_{[Z]}^{j_0, j_{s'}+1}(\mathcal{A}) \leq c_{|X_{A_1}|}(A_1) \cdots c_{|X_{A_q}|}(A_q).$$

However, since the number of  $Y$ 's is bounded by  $s$ , we have  $|X_{A_l}| \leq n_l + s$  for every  $1 \leq l \leq q$ . Moreover, since  $c_n(A_l)$  is an eventually nondecreasing function in  $n$  [9], we obtain

$$\dim_F P_{[Z]}^{j_0, j_{s'}+1}(\mathcal{A}) \leq c_{n_1+s}(A_1) \cdots c_{n_q+s}(A_q).$$

The remaining inequality follows from the fact that  $c_n(B) \simeq \Theta(n^t d^n)$  for PI algebras  $B$  as proved by Berele and Regev [5] for unital algebras and later by Giambruno and Zaicev [9] for arbitrary algebras, and thus in particular

$$\lim_{n \rightarrow \infty} \frac{c_{n+s}(A_l)}{c_n(A_l)} = K_2.$$

for some constant  $K_2 \in \mathbb{R}$ . □

**Theorem 3.13** (Upper bound). *There is a constant  $C$  such that*

$$c_n(A) \leq C \cdot n^{\frac{q-d}{2}+s} d^n.$$

*Proof.* By part (5) of lemma 3.12 it follows that

$$\dim_F P_{[Z]}^{j_0, j_{s'}+1}(\mathcal{A}) \leq C_1 \cdot c_{n_1}(A_1) \cdots c_{n_q}(A_q),$$

where  $n_1, \dots, n_q$  are determined by the path corresponding to  $[Z]$ . Combining this with lemma 3.11 results in:

$$\dim_F P_{[Z]}(\mathcal{A}) \leq C_2 \cdot c_{n_1}(A_1) \cdots c_{n_q}(A_q).$$

By lemma 3.8 it follows that

$$\sum_{[Z] \in \mathbf{Mon}_{[\vec{p}]}(n_1, \dots, n_q) / \sim} \dim_F P_{[Z]}(\mathcal{A}) \leq C_3 \cdot n^{s'} \binom{n-s'}{n_1, \dots, n_q} \cdot c_{n_1}(A_1) \cdots c_{n_q}(A_q),$$

where  $s' = n - n_1 - \dots - n_q$ . Thus,

$$\dim_F P_{[\vec{p}]}(\mathcal{A}) \leq C_4 \cdot n^{s'} \cdot \sum_{n_1 + \dots + n_q = n-s'} \binom{n-s'}{n_1, \dots, n_q} c_{n_1}(A_1) \cdots c_{n_q}(A_q).$$

By lemma 3.4 we obtain

$$c_n(A) \leq C_5 \cdot \sum_{s'=0}^s \left( n^{s'} \cdot \sum_{n_1+\dots+n_q=n-s'} \binom{n-s'}{n_1, \dots, n_q} c_{n_1}(A_1) \cdots c_{n_q}(A_q) \right).$$

Next, by [12],

$$\sum_{n_1+\dots+n_q=n-s'} \binom{n-s'}{n_1, \dots, n_q} c_{n_1}(A_1) \cdots c_{n_q}(A_q) \leq C_6 \cdot \sum_{n_1+\dots+n_q=n-s'} \binom{n-s'}{n_1, \dots, n_q} n_1^{\frac{1-d_1^2}{2}} d_1^{2n_1} \cdots n_q^{\frac{1-d_q^2}{2}} d_q^{2n_q},$$

for some constant  $C_6$ .

The last expression can be rewritten as

$$C_6 \cdot d^{n-s'} \sum_{n_1+\dots+n_q=n-s'} \binom{n-s'}{n_1, \dots, n_q} n_1^{\frac{1-d_1^2}{2}} \cdots n_q^{\frac{1-d_q^2}{2}} \cdot \left( \left( \frac{d_1}{d} \right)^{2n_1} \cdots \left( \frac{d_q}{d} \right)^{2n_q} \right),$$

which is by a theorem of Regev and Beckner (see below) asymptotically equal to

$$C_7 \cdot (n-s')^{\frac{q-d}{2}} d^{n-s'}.$$

All in all,

$$c_n(A) \leq C \cdot n^{\frac{q-d}{2}+s} d^n.$$

□

**Theorem 3.14.** (Regev and Bekner [2]) Let  $p = (p_1, \dots, p_q) \in \mathbb{Q}^q$  such that  $\sum p_i = 1$ , and suppose  $F(x_1, \dots, x_q)$  is a continuous homogeneous function of degree  $r$  with  $0 < F(p) < \infty$ . Then for  $\rho = r - \frac{1}{2}(\beta - 1)(q - 1)$  and  $\beta > 0$

$$\begin{aligned} \sum_{\substack{n_1 + \dots + n_q = n \\ n_i \neq 0}} \left[ \binom{n}{n_1, \dots, n_q} p_1^{n_1} \cdots p_q^{n_q} \right]^\beta F(n_1, \dots, n_q) \\ \sim n^\rho \beta^{-\frac{q-1}{2}} \left( \frac{1}{\sqrt{2\pi}} \right)^{(\beta-1)(q-1)} F(p) \left( \prod_{j=1}^q p_j \right)^{\frac{1-\beta}{2}} \end{aligned}$$

### 3.2 Lower bound

Let  $A$  be a basic algebra over a field  $F$ . Write  $A = (A_{ss} = A_1 \times \cdots \times A_q) \oplus J$  and denote by  $d_i^2$  the dimension of  $A_i$ . Furthermore,  $d = d_1^2 + \cdots + d_q^2$  and  $s$  denotes the nilpotency index of  $J$  minus 1.

*Remark 3.15.* In the sequel, as we may by linearity, all evaluations we consider of multilinear polynomials are from  $A_1 \cup \cdots \cup A_q \cup J$ .

Since  $A$  is basic, it possesses a multilinear polynomial  $f_0 = f_0(z_1, \dots, z_q; B = B_1 \cup \cdots \cup B_s; E)$ , where

1.  $|B_1| = \dots = |B_s| = d + 1$ .
2.  $f_0$  alternates on each one of the sets  $B_i$ . Therefore, any nonzero evaluation of the variables of  $f_0$  takes exactly one variable of every  $B_i$  to a radical element and the remaining variables (including the  $z$ 's and the ones from  $E$ ) to a semisimple element.
3. There is a nonzero evaluation of the variables of  $f_0$  such that  $\tilde{z}_i = 1(A_i)$  for  $i = 1, \dots, q$ .

Till the end we fix a nonzero evaluation which satisfies (3). Moreover, for  $i = 1, \dots, s$  we denote by  $w_i \in B_i$  the variable such that  $\tilde{w}_i \in J$ .

Consider the multilinear polynomial

$$f_1 = f_1(z_1, \dots, z_q; B; Y; E) = f_0(y_{1,1}y_{1,2}z_1y_{1,3}y_{1,4}, \dots, y_{q,1}y_{q,2}z_qy_{q,3}y_{q,4}; B; E),$$

where  $Y = \{y_{i,j} | i = 1, \dots, q, j = 1, \dots, 4\}$ . We shall abuse notation by writing  $f_1(z_1, \dots, z_q)$  (omitting  $E, Y$  and  $B$ ). Furthermore, We denote  $B \cup Y \cup E$  by  $BYE$ .

For every  $n$  denote  $X_n = \{x_1, \dots, x_n\}$ . Furthermore, consider a partition  $\mathbf{p}$  of the set  $X_n$  into  $q$  subsets denoted by  $X[A_1], \dots, X[A_q]$  such that all of them are nonempty (we need to assume here that  $n \geq q$ ).

Consider the symmetric groups  $S_{X[A_1]}, \dots, S_{X[A_q]}$  and their product  $S_{\mathbf{p}} = S_{X[A_1]} \times \dots \times S_{X[A_q]} < S_n$ . This group acts on the space  $P_{X_n; BYE}$  of all multilinear polynomials in  $X_n \cup BYE$  by

$$\sigma \cdot f(x_1, \dots, x_n; BYE) = f(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)}; BYE).$$

**Definition 3.16.** Let

$$P_{X_n; BYE}(A) = \frac{P_{X_n; BYE}}{P_{X_n; BYE} \cap id(A)}$$

be an  $S_n$ -module with the action induced from the  $S_n$ -action on  $P_{X_n; BYE}$ .

By the well known embedding of the relatively free algebra of  $A$  in  $U_A = A \otimes_F F(\theta_{i,j} : i \in \mathbb{N}, j = 1, \dots, \dim_F A)$  we may view the space  $P_{X_n; BYE}(A)$  as a subspace of  $U_A$ . This allows us to consider partial evaluation. To be more precise:

**Definition 3.17.** 1. For  $\tilde{\mathbf{j}}, \mathbf{j} \in \{1, \dots, d_1\} \times \dots \times \{1, \dots, d_q\}$ ,  $P_{X_n; BYE}^{\tilde{\mathbf{j}}, \mathbf{j}}(A) \subseteq U_A$  is the space obtained from  $P_{X_n; BYE}(A)$  by evaluating  $y_{i,2} \rightarrow \bar{y}_{i,2} = e_{\tilde{j}_i}(A_i)$  and  $y_{i,3} \rightarrow \bar{y}_{i,3} = e_{j_i}(A_i)$  for  $i = 1, \dots, q$ .

2. For  $g \in P_{X_n; BYE}(A)$  we denote by  $\bar{g}$  its image inside  $P_{X_n; BYE}^{\tilde{\mathbf{j}}, \mathbf{j}}(A)$ .

3. For  $i = 1, \dots, s$  and for any  $w \in B_i \setminus \{w_i\}$  we replace  $w$  by its value  $\tilde{w}$ . Note that  $\tilde{w}$  is necessary a semisimple element.

*Remark 3.18.* It is critical to notice that  $\dim_F P_{X_n; BYE}^{\tilde{\mathbf{j}}, \mathbf{j}}(A) \leq \dim_F P_{X_n; BYE}(A)$ . Therefore, for the lower bound it is enough to work with one of the spaces  $P_{X_n; BYE}^{\tilde{\mathbf{j}}, \mathbf{j}}(A)$ .



*Remark 3.19.* The elements of  $P_{X_n;BYE}^{\bar{\mathbf{j}},\mathbf{j}}(A)$ , as the elements of  $P_{X_n;BYE}(A)$ , will be referred as *multilinear polynomials*.

**Lemma 3.20.** *Consider a nonzero evaluation of  $\overline{f_1}$ . Then*

1. *The variables of  $\{z_1, \dots, z_q\} \cup Y \cup E$  are all evaluated by semisimple elements.*
2. *Each  $w_i \in B_i$ ,  $i = 1, \dots, s$ , is evaluated by a radical element (note the other elements of  $B_i$  were already replaced by semisimple elements).*
3. *For every  $i$  between 1 and  $q$ ,  $z_i$  is evaluated by an element of  $A_i$ .*

*Moreover, every nonzero evaluation of a polynomial  $\bar{g}_i$  by elements from  $A_i$  ( $i = 1, \dots, q$ ), such that  $e_{j_i} \bar{g}_i e_{\bar{j}_i} \neq 0$  is possible to extend to a nonzero evaluation of  $\overline{f_1}(g_1, \dots, g_q)$ .*

*Proof.* (1) and (2) are clear. (3) follows, since  $\bar{y}_{i,2}$  (and  $\bar{y}_{i,3}$ ) is an element from  $A_i$  and  $z_i$  must be a semisimple element (by (1)).

We turn to the last part of the Lemma. Since  $\bar{y}_{i,2} \tilde{g} \bar{y}_{i,3} \neq 0$ . We can find suitable evaluations of  $y_{i,1}$  and  $y_{i,4}$  so that the expression  $y_{i,1} \bar{y}_{i,2} \tilde{g} \bar{y}_{i,3} y_{i,4}$  is evaluated by any element of  $A_i$ . We are done since there is a nonzero evaluation of  $f_0$  with each  $z_i$  (as variable of  $f_0$ ) is evaluated by an element of  $A_i$ .  $\square$

**Lemma 3.21.** *Let  $V$  and  $W$  be finite dimensional vector spaces and let  $U$  be a subspace of  $\text{Hom}_F(V, W)$ . Fix a basis  $w_1, \dots, w_{\dim_F W}$  of  $W$ . Then there is an element  $\psi$  in the dual basis of  $W$  such that*

$$\dim_F(\psi \circ U) \geq \frac{\dim_F U}{\dim_F W}$$

*where  $\psi \circ U$  is the space of all elements  $\psi \circ T$  where  $T \in U$ .*

*Proof.* It is clear that the map  $\Psi : \text{Hom}_F(V, W) \rightarrow V^* \oplus \dots \oplus V^*$  ( $\dim_F W$  times) given by

$$\Psi(T) = (\psi_1 \circ T, \dots, \psi_{\dim_F W} \circ T),$$

is injective (in fact, isomorphism), where  $\psi_1, \dots, \psi_{\dim_F W}$  is any basis (in our case it is the dual basis) of  $W^*$ . As a result,

$$\dim_F U \leq \sum_{i=1}^{\dim_F W} \dim_F \psi_i \circ U.$$

So there is some  $i_0$  such that

$$\dim_F U \leq \dim_F W \cdot \dim_F \psi_{i_0} \circ U.$$

So  $\psi = \psi_{i_0}$  is the required dual basis element.  $\square$

Next, consider the spaces  $T_{X[A_i]}(A_i) = \text{Hom}_F(A_i^{\otimes n_i}, A_i)$  where  $i = 1, \dots, q$  ( $n_i$  is the number of elements of  $X[A_i]$  in the partition of  $n$ ). Recall the embedding of  $P_{X[A_i]}(A_i)$  into  $T_{X[A_i]}(A_i)$  given by

$$g \rightarrow g(a_1 \otimes \dots \otimes a_{n_i}) = g(a_1, \dots, a_{n_i}).$$

We apply the above lemma in the following setup:  $V = A_i^{\otimes n_i}$ ,  $W = A_i$  and  $U = P_{X[A_i]}(A_i) \subseteq T_{X[A_i]}(A_i)$ . We use the basis of elementary matrices as a basis of  $W$ . Denote by  $\psi_{\tilde{j}_1, j_1}, \dots, \psi_{\tilde{j}_q, j_q}$  the dual elements for  $A_i$  for any  $i = 1, \dots, q$ . Note that  $\psi_{\tilde{j}_i, j_i} : A_i \rightarrow F$  is the map assigning to each matrix from  $A_i$  its  $e_{\tilde{j}_i, j_i}$  coefficient. For the given  $\mathbf{j} = (j_1, \dots, j_q)$  and  $\tilde{\mathbf{j}} = (\tilde{j}_1, \dots, \tilde{j}_q)$  we simplify our notation and denote the space  $P_{X_n; BYE}^{\tilde{\mathbf{j}}, \mathbf{j}}(A)$  by  $\bar{P}_{X_n; BYE}(A)$ .

**Theorem 3.22.** *The mapping*

$$\phi_{\mathbf{p}} : P_{X[A_1]}(A_1) \otimes \dots \otimes P_{X[A_q]}(A_q) \rightarrow \bar{P}_{X_n; BYE}(A)$$

*given by*

$$\phi_{\mathbf{p}}(g_1 \otimes \dots \otimes g_q) = \overline{f_1(g_1, \dots, g_q)},$$

*is well defined.*

*Moreover, if we denote by  $M(\mathbf{p})$  the image of  $\phi_{\mathbf{p}}$ , then*

$$\dim M(\mathbf{p}) \geq \frac{1}{d_1^2 \dots d_q^2} \cdot c_{n_1}(A_1) \dots c_{n_q}(A_q),$$

*where  $n_1 = |X[A_1]|, \dots, n_q = |X[A_q]|$ .*

*Proof.* The map  $\phi_i : P_{X_i(A_i)}(A_i) \rightarrow \bar{P}_{X_n; BYE}(A)$  given by  $\phi_i(g) = \overline{f_1|_{z_i \rightarrow g_i}}$  is well defined, since in any nonzero evaluation of  $\overline{f_1}$  the variables of  $X[A_i]$  are all evaluated by elements of  $A_i$  (see lemma 3.20), so for  $g_i \in \text{id}(A_i)$  there is no nonzero evaluation of  $\overline{f_1|_{z_i \rightarrow g_i}}$ . In other words,

$$g_i \in \text{id}(A_i) \implies \phi_i(g) = 0.$$

Therefore, the map

$$\phi'_{\mathbf{p}} : P_{X[A_1]}(A_1) \times \dots \times P_{X[A_q]}(A_q) \rightarrow \bar{P}_{X_n; BYE}(A)$$

*given by*

$$\phi'_{\mathbf{p}}(g_1, \dots, g_q) = \overline{f_1(g_1, \dots, g_q)}$$

*is well defined. Since clearly  $\phi'_{\mathbf{p}}$  is multilinear, it induces the map*

$$\phi_{\mathbf{p}} : P_{X[A_1]}(A_1) \otimes \dots \otimes P_{X[A_q]}(A_q) \rightarrow \bar{P}_{X_n; BYE}(A).$$

Suppose  $\psi_{\tilde{j}_i, j_i} \circ g_{(A_i, 1)}, \dots, \psi_{\tilde{j}_i, j_i} \circ g_{(A_i, t_i)} \in (A_i^{\otimes n_i})^*$  is a basis for  $\psi_{\tilde{j}_i, j_i} \circ P_{X[A_i]}(A_i)$  (this is an abuse of language, it is in fact a dual basis of a quotient of  $A_i^{\otimes n_i}$  lifted to  $A_i^{\otimes n_i}$ ).

Applying the lemma it is clear now that in order to complete the proof it is enough to prove that  $\phi_{\mathbf{p}}$  is injective when restricted to the subspace  $T$  spanned by  $g_{(A_1, \alpha_1)} \otimes \cdots \otimes g_{(A_q, \alpha_q)}$ , where  $\alpha_1 = 1, \dots, t_1; \dots; \alpha_q = 1, \dots, t_q$ .

Suppose there are scalars  $c_{\alpha_1, \dots, \alpha_q} \in F$  such that

$$0 = \sum_{\alpha_1, \dots, \alpha_q} c_{\alpha_1, \dots, \alpha_q} \phi_{\mathbf{p}}(g_{(A_1, \alpha_1)} \otimes \cdots \otimes g_{(A_q, \alpha_q)}) = \sum_{\alpha_1, \dots, \alpha_q} c_{\alpha_1, \dots, \alpha_q} \overline{f_1(g_{(A_1, \alpha_1)}, \dots, g_{(A_q, \alpha_q)})}.$$

Denote  $\mathbf{a}_k^{(A_1)} = \sum_l a_1(l) \otimes \cdots \otimes a_{n_1}(l)$ . For each  $l$  we consider the substitution  $x_1 \rightarrow a_1(l), \dots, x_{n_1} \rightarrow a_{n_1}(l)$ , where (without loss of generality)  $X[A_1] = \{x_1, \dots, x_{n_1}\}$ , obtaining  $\overline{f_1(g_{(A_1, \alpha_1)}, \dots, g_{(A_q, \alpha_q)})}(l)$ . Hence,

$$\begin{aligned} \sum_l \overline{f_1(g_{(A_1, \alpha_1)}, \dots, g_{(A_q, \alpha_q)})}(l) &= \sum_{\alpha_1, \dots, \alpha_q} c_{\alpha_1, \dots, \alpha_q} \overline{f_1(g_{(A_1, \alpha_1)}(\mathbf{a}_1^{(A_1)}), g_{(A_2, \alpha_2)}, \dots, g_{(A_q, \alpha_q)})} \\ &= \sum_{\alpha_1=1, \alpha_2, \dots, \alpha_q} c_{\alpha_1=1, \alpha_2, \dots, \alpha_q} \overline{f_1(e_{\tilde{j}_1, j_1}(A_1), g_{(A_2, \alpha_2)}, \dots, g_{(A_q, \alpha_q)})} \end{aligned}$$

By considering  $\mathbf{a}_1^{(A_2)}, \dots, \mathbf{a}_1^{(A_q)}$  and applying the same argument on the last expression we conclude that

$$c_{1, \dots, 1} \cdot \overline{f_1(e_{\tilde{j}_1, j_1}(A_1), \dots, e_{\tilde{j}_q, j_q}(A_q))} = 0.$$

By 3.20, it follows that  $\overline{f_1(e_{\tilde{j}_1, j_1}(A_1), \dots, e_{\tilde{j}_q, j_q}(A_q))} \neq 0$ , hence  $c_{1, \dots, 1} = 0$ .

It is clear, that the same argument will work on every  $\mathbf{a}_{\alpha_1}^{(A_1)}, \dots, \mathbf{a}_{\alpha_q}^{(A_q)}$ , so every  $c_{\alpha_1, \dots, \alpha_q} = 0$ .  $\square$

Next we study the connection between the different  $M(\mathbf{p})$ .

**Lemma 3.23.** *Fix some  $\mathbf{p}_0 = (X[A_1], \dots, X[A_q])$ ,  $n_i$  is the number of elements in  $X[A_i]$ . Let  $\mathcal{T} = \{e = \tau_1, \tau_2, \dots, \tau_l\}$  be a transversal of  $S_{\mathbf{p}_0}$  in  $S_n$ . Then, the sum*

$$M_{\text{tot}}(\vec{n}) := M(\tau_1 \mathbf{p}_0) + \cdots + M(\tau_l \mathbf{p}_0),$$

where  $\vec{n} = (n_1, \dots, n_q)$ , is direct.

Note that  $M_{\text{tot}}(\vec{n}) = \overline{FS_{\mathbf{p}_0} \cdot f_1(X_n; w_1, \dots, w_s)}$ .

*Remark 3.24.*  $l = \frac{n!}{n_1! \cdots n_q!} = \binom{n}{n_1, \dots, n_q}$ .

*Proof.* Suppose

$$0 = \sum_{k=1}^l \alpha_k h_k,$$

where  $0 \neq h_k \in M(\tau_k \mathbf{p}_0)$  and  $\alpha_k \in F$ . Since

$$h_1 = \sum_{\sigma \in S_{\mathbf{p}_0}} \beta_{\sigma} f_1(x_{\sigma(1)} \cdots x_{\sigma(n_1)}, \dots, x_{\sigma(n_1 + \cdots + n_{q-1} + 1)} \cdots x_{\sigma(n)}) \neq 0$$

we obtain by Lemma 3.20 a nonzero evaluation  $\xi : F\{X_n, BYE\} \rightarrow A$  which sends the variables of  $X[A_i]$  to elements of  $A_i$  (for  $i = 1, \dots, q$ ). We claim this evaluation sends each  $h_k$  ( $k \neq 1$ ) to zero. Indeed, at least one of the sets  $\tau_k X[A_1], \dots, \tau_k X[A_q]$  must have an element  $x \in \tau_{i_0} X[A_{i_0}]$  which is sent to some  $A_i$ ,  $i \neq i_0$  and hence,

$$h_k = \overline{\sum_{\sigma \in S_{p_0}} \beta_{\tau_k \sigma} f_1(x_{\tau_k \sigma(1)} \cdots x_{\tau_k \sigma(n_1)}, \dots, x_{\tau_k \sigma(n_1 + \dots + n_{q-1} + 1)} \cdots x_{\tau_k \sigma(n)})}$$

is sent to zero. It follows that  $0 = \alpha_1 \cdot \xi(h_1)$  and hence  $\alpha_1 = 0$ .

Repeating this argument for  $h_2, \dots, h_l$  yields  $\alpha_2 = \alpha_3 = \dots = \alpha_l = 0$ .  $\square$

Write

$$M_{\text{tot}}(n) := \sum_{n_1 + \dots + n_l = n} M_{\text{tot}}(\vec{n} = (n_1, \dots, n_l)).$$

Note that in view of our notation above for  $M_{\text{tot}}(\vec{n})$  we have  $M_{\text{tot}}(n) = \overline{FS_n \cdot f_1(X_n; w_1, \dots, w_s)}$ .

Using a similar argument as in the previous lemma we obtain the following

**Lemma 3.25.** *For every  $n$ , the sum appearing in  $M_{\text{tot}}(n)$  is direct.*

Consider now the group  $S_{X_n \cup \{w_1, \dots, w_s\}}$  (see the beginning of the section for the definition of  $w_1, \dots, w_s$ ) and let  $\{e = \sigma_1, \sigma_2, \dots, \sigma_u\}$  be a transversal of  $S_{X_n}$  inside the aforementioned group.

*Remark 3.26.*  $u = \frac{(n+s)!}{n!} = \Theta(n^s)$ .

**Lemma 3.27.** *The sum*

$$M_{\text{total}}(n) = \sum_{k=1}^u \sigma_k M_{\text{tot}}(n)$$

*is direct, where*

$$\sigma_k M_{\text{tot}}(n) = \overline{\sigma_k FS_n \cdot f_1(X_n; w_1, \dots, w_s)}.$$

*Proof.* The idea used in the proof of lemma 3.23 works also here. Suppose

$$0 = \sum_{k=1}^u \alpha_k h_k,$$

where  $0 \neq h_k \in \sigma_k M_{\text{tot}}(n)$  and  $\alpha_k \in F$ . Since  $h_1 \neq 0$ , we obtain by lemma 3.20, a nonzero evaluation  $\xi : F\{X_n, BYE\} \rightarrow A$  which sends the variables in  $X_n$  to elements of  $A_{ss}$  and  $w_1, \dots, w_q$  to elements of  $J$ . This evaluation sends each  $h_k$  ( $k \neq 1$ ) to zero. Indeed,  $\sigma_k B_1, \dots, \sigma_k B_s$  are alternating sets of cardinality  $d+1$  in  $h_k$ . However, there is some  $i_0$  for which  $\sigma_k B_{i_0}$  does not contain any of the variables  $w_1, \dots, w_s$ . Hence  $\xi(\sigma_k B_{i_0}) \cap J = \emptyset$ . As a result,  $\xi(h_k) = 0$ . Thus,  $0 = \alpha_1 \xi(h_1)$  and so  $\alpha_1 = 0$ .

Repeating this argument to  $h_2, \dots, h_l$  yields  $\alpha_2 = \dots = \alpha_l = 0$ .  $\square$

Now we show that asymptotically  $\dim_F M_{\text{tot}}(n)$  has the desired lower bound. Recall that  $c_{n_i}(A_i) \sim K_i n^{\frac{1-d_i^2}{2}} d_i^{2n}$  for some constant  $K_i \in \mathbb{R}$  [12]. As a result of lemma 3.23, lemma 3.25 and the above lemma we get (the inequality  $\lesssim$  is asymptotically)

$$\sum_{n_1+\dots+n_q=n} \frac{K_1 \cdots K_q}{d_1^2 \cdots d_q^2} \cdot \binom{n}{n_1, \dots, n_q} \cdot n_1^{\frac{1-d_1^2}{2}} \cdots n_q^{\frac{1-d_q^2}{2}} \cdot d_1^{2n_1} \cdots d_q^{2n_q} \cdot n^s \lesssim \dim_F M_{\text{total}}(n).$$

This can be rewritten as

$$C_1 d^n n^s \cdot \sum_{\substack{(n_1, \dots, n_q) \\ n_1 + \dots + n_q = n}} \left( \binom{n}{n_1, \dots, n_q} \cdot p_1^{n_1} \cdots p_q^{n_q} \right) \cdot F(n_1, \dots, n_q),$$

where  $p_1 = \frac{d_1^2}{d}, \dots, p_q = \frac{d_q^2}{d}$ ,  $d = d_1^2 + \dots + d_q^2$ , and  $F(x_1, \dots, x_q) = x_1^{\frac{1-d_1^2}{2}} \cdots x_q^{\frac{1-d_q^2}{2}}$ .

Finally we apply once again Theorem 3.14. We obtain

$$\dim_F M_{\text{total}}(n) \geq C n^{\frac{q-d}{2}+s} d^n$$

(here  $C$  is some nonnegative constant).

**Corollary 3.28** (Lower bound). *Let  $A = A_1 \times \cdots \times A_q \oplus J(A)$  be a basic algebra of Kemer index  $\kappa_A = (d, s)$ . Then*

$$c_n(A) \geq C n^{\frac{q-d}{2}+s} d^n.$$

for some constant  $C \in \mathbb{R}$ .

*Proof.* It is enough to show that

$$\dim_F c_{n+\gamma}(A) \geq \dim_F M_{\text{total}}(n),$$

for some  $\gamma$  independent of  $n$ . Indeed,

$$M_{\text{total}}(n) \subseteq \overline{P_{X_n; BYE}(A)}$$

and  $\overline{P_{X_n; BYE}(A)}$  is a projection of  $P_{X_n; BYE}(A)$ . Hence, the statement stands for  $\gamma = |BYE|$ .  $\square$

So combined with Theorem 3.13 we finally get a positive answer on Giambruno's conjecture.

**Theorem 3.29.** *Let  $A = A_1 \times \cdots \times A_q \oplus J(A)$  be a basic algebra of Kemer index  $\kappa_A = (d, s)$ . Then*

$$c_n(A) \simeq C n^{\frac{q-d}{2}+s} d^n.$$

for some constant  $0 < C \in \mathbb{R}$ .

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